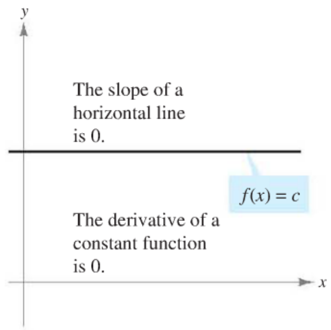
3.2 Basic Differentiation Rules and Rates of Change

- **Find the derivative of a function using the Constant Rule.**
- **Find the derivative of a function using the Power Rule.**
- **Find the derivative of a function using the Constant Multiple Rule.**
- **Find the derivative of a function using the Sum and Difference Rules.**
- **Find the derivatives of the sine, cosine, and exponential functions.**
- **Use derivatives to find rates of change.**

The Constant Rule

In Section 3.1 you used the limit definition to find derivatives. In this and the next two sections you will be introduced to several "differentiation rules" that allow you to find derivatives without the *direct* use of the limit definition.



Notice that the Constant Rule is equivalent to saying that the slope of a horizontal line is 0. This demonstrates the relationship between slope and derivative.
Figure 3.14

THEOREM 3.2 THE CONSTANT RULE

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$

(See Figure 3.14.)

PROOF Let $f(x) = c$. Then, by the limit definition of the derivative,

$$\begin{aligned}\frac{d}{dx}[c] &= f'(x)\\ &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x}\\ &= \lim_{\Delta x \to 0} 0\\ &= 0.\end{aligned}$$

EXAMPLE 1 Using the Constant Rule

Function	*Derivative*
a. $y = 7$	$dy/dx = 0$
b. $f(x) = 0$	$f'(x) = 0$
c. $s(t) = -3$	$s'(t) = 0$
d. $y = k\pi^2$, k is constant	$y' = 0$

EXPLORATION

Writing a Conjecture Use the definition of the derivative given in Section 3.1 to find the derivative of each of the following. What patterns do you see? Use your results to write a conjecture about the derivative of $f(x) = x^n$.

a. $f(x) = x^1$ **b.** $f(x) = x^2$ **c.** $f(x) = x^3$
d. $f(x) = x^4$ **e.** $f(x) = x^{1/2}$ **f.** $f(x) = x^{-1}$

The Power Rule

Before proving the next rule, review the procedure for expanding a binomial.

$$(x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$
$$(x + \Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$
$$(x + \Delta x)^4 = x^4 + 4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4$$

The general binomial expansion for a positive integer n is

$$(x + \Delta x)^n = x^n + nx^{n-1}(\Delta x) + \underbrace{\frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n}_{(\Delta x)^2 \text{ is a factor of these terms.}}.$$

This binomial expansion is used in proving a special case of the Power Rule.

NOTE From Example 7 in Section 3.1, you know that the function $f(x) = x^{1/3}$ is defined at $x = 0$, but is not differentiable at $x = 0$. This is because $x^{-2/3}$ is not defined on an interval containing 0.

THEOREM 3.3 THE POWER RULE

If n is a real number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.

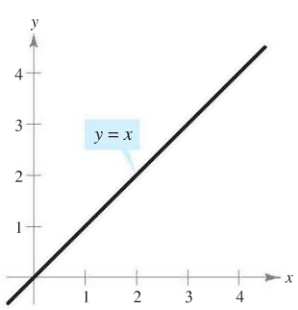
PROOF If n is a positive integer greater than 1, then the binomial expansion produces the following.

$$\frac{d}{dx}[x^n] = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}(\Delta x) + \frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n - x^n}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)x^{n-2}}{2}(\Delta x) + \cdots + (\Delta x)^{n-1} \right]$$
$$= nx^{n-1} + 0 + \cdots + 0$$
$$= nx^{n-1}.$$

This proves the case for which n is a positive integer greater than 1. It is left to you to prove the case for $n = 1$. Example 7 in Section 3.3 proves the case for which n is a negative integer. The cases for which n is rational and n is irrational are left as an exercise (see Section 3.5, Exercise 100). ■

When using the Power Rule, the case for which $n = 1$ is best thought of as a separate differentiation rule. That is,

$$\frac{d}{dx}[x] = 1. \qquad \text{Power Rule when } n = 1$$



The slope of the line $y = x$ is 1.
Figure 3.15

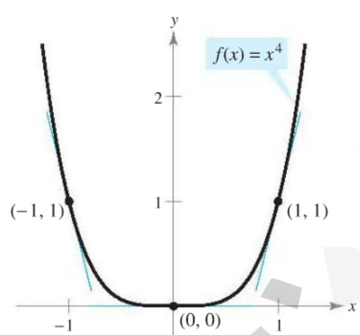
This rule is consistent with the fact that the slope of the line $y = x$ is 1, as shown in Figure 3.15.

EXAMPLE 2 Using the Power Rule

Function	*Derivative*
a. $f(x) = x^3$	$f'(x) = 3x^2$
b. $g(x) = \sqrt[3]{x}$	$g'(x) = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$
c. $y = \frac{1}{x^2}$	$\frac{dy}{dx} = \frac{d}{dx}[x^{-2}] = (-2)x^{-3} = -\frac{2}{x^3}$

In Example 2(c), note that *before* differentiating, $1/x^2$ was rewritten as x^{-2}. Rewriting is the first step in *many* differentiation problems.

Given: $y = \frac{1}{x^2}$ ⇨ Rewrite: $y = x^{-2}$ ⇨ Differentiate: $\frac{dy}{dx} = (-2)x^{-3}$ ⇨ Simplify: $\frac{dy}{dx} = -\frac{2}{x^3}$



The slope of a graph at a point is the value of the derivative at that point.
Figure 3.16

EXAMPLE 3 Finding the Slope of a Graph

Find the slope of the graph of $f(x) = x^4$ when

a. $x = -1$ **b.** $x = 0$ **c.** $x = 1$.

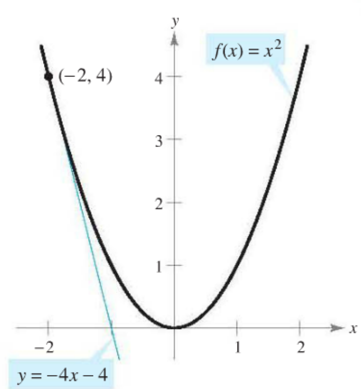
Solution The derivative of f is $f'(x) = 4x^3$.

a. When $x = -1$, the slope is $f'(-1) = 4(-1)^3 = -4$. Slope is negative.

b. When $x = 0$, the slope is $f'(0) = 4(0)^3 = 0$. Slope is zero.

c. When $x = 1$, the slope is $f'(1) = 4(1)^3 = 4$. Slope is positive.

In Figure 3.16, note that the slope of the graph is negative at the point $(-1, 1)$, the slope is zero at the point $(0, 0)$, and the slope is positive at the point $(1, 1)$.



The line $y = -4x - 4$ is tangent to the graph of $f(x) = x^2$ at the point $(-2, 4)$.
Figure 3.17

EXAMPLE 4 Finding an Equation of a Tangent Line

Find an equation of the tangent line to the graph of $f(x) = x^2$ when $x = -2$.

Solution To find the *point* on the graph of f, evaluate the original function at $x = -2$.

$$(-2, f(-2)) = (-2, 4) \qquad \text{Point on graph}$$

To find the *slope* of the graph when $x = -2$, evaluate the derivative, $f'(x) = 2x$, at $x = -2$.

$$m = f'(-2) = -4 \qquad \text{Slope of graph at } (-2, 4)$$

Now, using the point-slope form of the equation of a line, you can write

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$
$$y - 4 = -4[x - (-2)] \qquad \text{Substitute for } y_1, m, \text{ and } x_1.$$
$$y = -4x - 4. \qquad \text{Simplify.}$$

(See Figure 3.17.)

The Constant Multiple Rule

THEOREM 3.4 THE CONSTANT MULTIPLE RULE

If f is a differentiable function and c is a real number, then cf is also differentiable and $\frac{d}{dx}[cf(x)] = cf'(x)$.

PROOF

$$\frac{d}{dx}[cf(x)] = \lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} \qquad \text{Definition of derivative}$$

$$= \lim_{\Delta x \to 0} c\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= c\left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right] \qquad \text{Apply Theorem 2.2.}$$

$$= cf'(x)$$

■

Informally, the Constant Multiple Rule states that constants can be factored out of the differentiation process, even if the constants appear in the denominator.

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] = cf'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{c}\right] = \frac{d}{dx}\left[\left(\frac{1}{c}\right)f(x)\right]$$

$$= \left(\frac{1}{c}\right)\frac{d}{dx}[f(x)] = \left(\frac{1}{c}\right)f'(x)$$

EXAMPLE 5 Using the Constant Multiple Rule

Function	*Derivative*
a. $y = \frac{2}{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{-1}] = 2\frac{d}{dx}[x^{-1}] = 2(-1)x^{-2} = -\frac{2}{x^2}$
b. $f(t) = \frac{4t^2}{5}$	$f'(t) = \frac{d}{dt}\left[\frac{4}{5}t^2\right] = \frac{4}{5}\frac{d}{dt}[t^2] = \frac{4}{5}(2t) = \frac{8}{5}t$
c. $y = 2\sqrt{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{1/2}] = 2\left(\frac{1}{2}x^{-1/2}\right) = x^{-1/2} = \frac{1}{\sqrt{x}}$
d. $y = \frac{1}{2\sqrt[3]{x^2}}$	$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}x^{-2/3}\right] = \frac{1}{2}\left(-\frac{2}{3}\right)x^{-5/3} = -\frac{1}{3x^{5/3}}$
e. $y = -\frac{3x}{2}$	$y' = \frac{d}{dx}\left[-\frac{3}{2}x\right] = -\frac{3}{2}(1) = -\frac{3}{2}$

■

The Constant Multiple Rule and the Power Rule can be combined into one rule. The combination rule is

$$\frac{d}{dx}[cx^n] = cnx^{n-1}.$$

EXAMPLE 6 Using Parentheses When Differentiating

	Original Function	*Rewrite*	*Differentiate*	*Simplify*
a.	$y = \frac{5}{2x^3}$	$y = \frac{5}{2}(x^{-3})$	$y' = \frac{5}{2}(-3x^{-4})$	$y' = -\frac{15}{2x^4}$
b.	$y = \frac{5}{(2x)^3}$	$y = \frac{5}{8}(x^{-3})$	$y' = \frac{5}{8}(-3x^{-4})$	$y' = -\frac{15}{8x^4}$
c.	$y = \frac{7}{3x^{-2}}$	$y = \frac{7}{3}(x^2)$	$y' = \frac{7}{3}(2x)$	$y' = \frac{14x}{3}$
d.	$y = \frac{7}{(3x)^{-2}}$	$y = 63(x^2)$	$y' = 63(2x)$	$y' = 126x$

■

The Sum and Difference Rules

THEOREM 3.5 THE SUM AND DIFFERENCE RULES

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \qquad \text{Difference Rule}$$

PROOF A proof of the Sum Rule follows from Theorem 2.2. (The Difference Rule can be proved in a similar way.)

$$\begin{aligned}
\frac{d}{dx}[f(x) + g(x)] &= \lim_{\Delta x \to 0} \frac{[f(x + \Delta x) + g(x + \Delta x)] - [f(x) + g(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x)
\end{aligned}$$

■

EXPLORATION

Use a graphing utility to graph the function

$$f(x) = \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$$

for $\Delta x = 0.01$. What does this function represent? Compare this graph with that of the cosine function. What do you think the derivative of the sine function equals?

The Sum and Difference Rules can be extended to any finite number of functions. For instance, if $F(x) = f(x) + g(x) - h(x)$, then $F'(x) = f'(x) + g'(x) - h'(x)$.

EXAMPLE 7 Using the Sum and Difference Rules

	Function	*Derivative*
a.	$f(x) = x^3 - 4x + 5$	$f'(x) = 3x^2 - 4$
b.	$g(x) = -\frac{x^4}{2} + 3x^3 - 2x$	$g'(x) = -2x^3 + 9x^2 - 2$

■

FOR FURTHER INFORMATION For the outline of a geometric proof of the derivatives of the sine and cosine functions, see the article "The Spider's Spacewalk Derivation of sin′ and cos′" by Tim Hesterberg in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

Derivatives of Sine and Cosine Functions

In Section 2.3, you studied the following limits.

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0$$

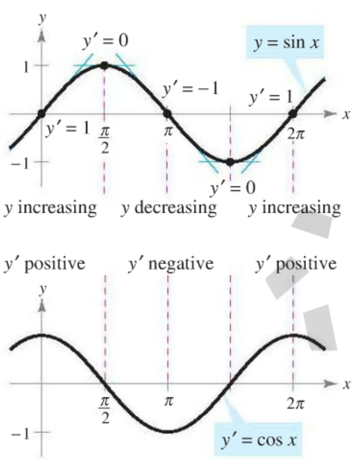
These two limits can be used to prove differentiation rules for the sine and cosine functions. (The derivatives of the other four trigonometric functions are discussed in Section 3.3.)

THEOREM 3.6 DERIVATIVES OF SINE AND COSINE FUNCTIONS

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

PROOF

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x - (\sin x)(1 - \cos \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[(\cos x)\left(\frac{\sin \Delta x}{\Delta x}\right) - (\sin x)\left(\frac{1 - \cos \Delta x}{\Delta x}\right)\right] \\
&= \cos x\left(\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}\right) - \sin x\left(\lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x}\right) \\
&= (\cos x)(1) - (\sin x)(0) \\
&= \cos x
\end{aligned}$$



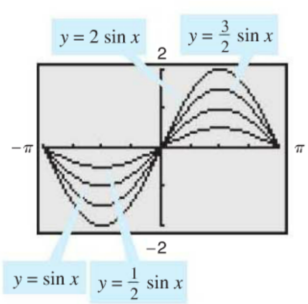
The derivative of the sine function is the cosine function.
Figure 3.18

This differentiation rule is shown graphically in Figure 3.18. Note that for each x, the *slope* of the sine curve is equal to the value of the cosine. The proof of the second rule is left as an exercise (see Exercise 124). ■

EXAMPLE 8 Derivatives Involving Sines and Cosines

Function	*Derivative*
a. $y = 2 \sin x$	$y' = 2 \cos x$
b. $y = \dfrac{\sin x}{2} = \dfrac{1}{2} \sin x$	$y' = \dfrac{1}{2} \cos x = \dfrac{\cos x}{2}$
c. $y = x + \cos x$	$y' = 1 - \sin x$

■



$\frac{d}{dx}[a \sin x] = a \cos x$
Figure 3.19

TECHNOLOGY A graphing utility can provide insight into the interpretation of a derivative. For instance, Figure 3.19 shows the graphs of

$$y = a \sin x$$

for $a = \frac{1}{2}, 1, \frac{3}{2}$, and 2. Estimate the slope of each graph at the point $(0, 0)$. Then verify your estimates analytically by evaluating the derivative of each function when $x = 0$.

EXPLORATION

Use a graphing utility to graph the function

$$f(x) = \frac{e^{x+\Delta x} - e^x}{\Delta x}$$

for $\Delta x = 0.01$. What does this function represent? Compare this graph with that of the exponential function. What do you think the derivative of the exponential function equals?

Derivatives of Exponential Functions

One of the most intriguing (and useful) characteristics of the natural exponential function is that *it is its own derivative*. Consider the following.

Let $f(x) = e^x$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x}$$

The definition of e

$$\lim_{\Delta x \to 0} (1 + \Delta x)^{1/\Delta x} = e$$

tells you that for small values of Δx, you have $e \approx (1 + \Delta x)^{1/\Delta x}$, which implies that $e^{\Delta x} \approx 1 + \Delta x$. Replacing $e^{\Delta x}$ by this approximation produces the following.

$$f'(x) = \lim_{\Delta x \to 0} \frac{e^x[e^{\Delta x} - 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^x[(1 + \Delta x) - 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^x \Delta x}{\Delta x}$$

$$= e^x$$

STUDY TIP The key to the formula for the derivative of $f(x) = e^x$ is the limit

$$\lim_{x \to 0} (1 + x)^{1/x} = e.$$

This important limit was introduced on page 51 and formalized later on page 85. It is used to conclude that for $\Delta x \approx 0$,

$$(1 + \Delta x)^{1/\Delta x} \approx e.$$

This result is stated in the next theorem.

THEOREM 3.7 DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

$$\frac{d}{dx}[e^x] = e^x$$

You can interpret Theorem 3.7 graphically by saying that the slope of the graph of $f(x) = e^x$ at any point (x, e^x) is equal to the y-coordinate of the point, as shown in Figure 3.20.

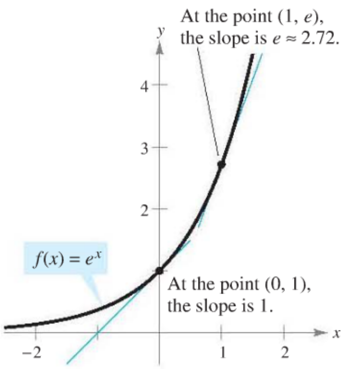


Figure 3.20

EXAMPLE 9 Derivatives of Exponential Functions

Find the derivative of each function.

a. $f(x) = 3e^x$ **b.** $f(x) = x^2 + e^x$ **c.** $f(x) = \sin x - e^x$

Solution

a. $f'(x) = 3\frac{d}{dx}[e^x] = 3e^x$

b. $f'(x) = \frac{d}{dx}[x^2] + \frac{d}{dx}[e^x] = 2x + e^x$

c. $f'(x) = \frac{d}{dx}[\sin x] - \frac{d}{dx}[e^x] = \cos x - e^x$ ■

Rates of Change

You have seen how the derivative is used to determine slope. The derivative can also be used to determine the rate of change of one variable with respect to another. Applications involving rates of change occur in a wide variety of fields. A few examples are population growth rates, production rates, water flow rates, velocity, and acceleration.

A common use for rate of change is to describe the motion of an object moving in a straight line. In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion. On such lines, movement to the right (or upward) is considered to be in the positive direction, and movement to the left (or downward) is considered to be in the negative direction.

The function s that gives the position (relative to the origin) of an object as a function of time t is called a **position function.** If, over a period of time Δt, the object changes its position by the amount $\Delta s = s(t + \Delta t) - s(t)$, then, by the familiar formula

$$\text{Rate} = \frac{\text{distance}}{\text{time}}$$

the **average velocity** is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t}.$$

Average velocity

EXAMPLE 10 Finding Average Velocity of a Falling Object

If a billiard ball is dropped from a height of 100 feet, its height s at time t is given by the position function

$$s = -16t^2 + 100$$

Position function

where s is measured in feet and t is measured in seconds. Find the average velocity over each of the following time intervals.

a. $[1, 2]$ **b.** $[1, 1.5]$ **c.** $[1, 1.1]$

Solution

a. For the interval $[1, 2]$, the object falls from a height of $s(1) = -16(1)^2 + 100 = 84$ feet to a height of $s(2) = -16(2)^2 + 100 = 36$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{36 - 84}{2 - 1} = \frac{-48}{1} = -48 \text{ feet per second.}$$

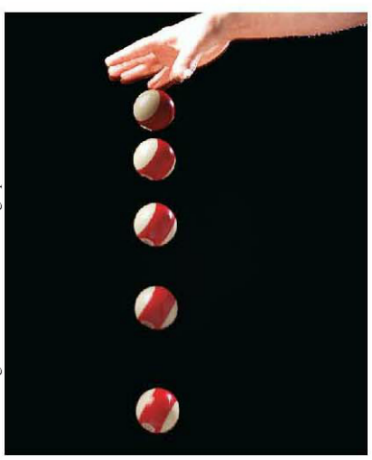
b. For the interval $[1, 1.5]$, the object falls from a height of 84 feet to a height of 64 feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{64 - 84}{1.5 - 1} = \frac{-20}{0.5} = -40 \text{ feet per second.}$$

c. For the interval $[1, 1.1]$, the object falls from a height of 84 feet to a height of 80.64 feet. The average velocity is

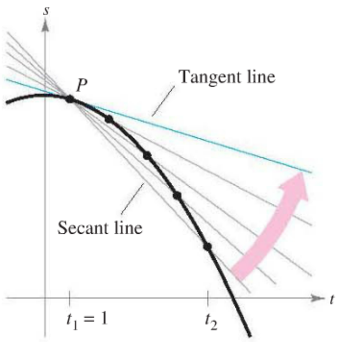
$$\frac{\Delta s}{\Delta t} = \frac{80.64 - 84}{1.1 - 1} = \frac{-3.36}{0.1} = -33.6 \text{ feet per second.}$$

Note that the average velocities are *negative*, indicating that the object is moving downward.



Richard Megna/Fundamental Photographs

Time-lapse photograph of a free-falling billiard ball



The average velocity between t_1 and t_2 is the slope of the secant line, and the instantaneous velocity at t_1 is the slope of the tangent line.
Figure 3.21

Suppose that in Example 10 you wanted to find the *instantaneous* velocity (or simply the velocity) of the object when $t = 1$. Just as you can approximate the slope of the tangent line by calculating the slope of the secant line, you can approximate the velocity at $t = 1$ by calculating the average velocity over a small interval $[1, 1 + \Delta t]$ (see Figure 3.21). By taking the limit as Δt approaches zero, you obtain the velocity when $t = 1$. Try doing this—you will find that the velocity when $t = 1$ is -32 feet per second.

In general, if $s = s(t)$ is the position function for an object moving along a straight line, the **velocity** of the object at time t is

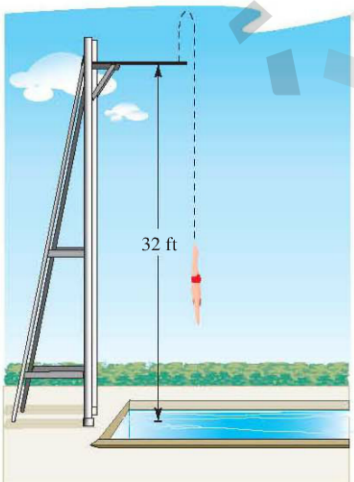
$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t).$$ Velocity function

In other words, the velocity function is the derivative of the position function. Velocity can be negative, zero, or positive. The **speed** of an object is the absolute value of its velocity. Speed cannot be negative.

The position of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation

$$s(t) = \frac{1}{2}gt^2 + v_0t + s_0$$ Position function

where s_0 is the initial height of the object, v_0 is the initial velocity of the object, and g is the acceleration due to gravity. On Earth, the value of g is approximately -32 feet per second per second or -9.8 meters per second per second.



Velocity is positive when an object is rising, and is negative when an object is falling. Notice that the diver moves upward for the first half-second because the velocity is positive for $0 < t < \frac{1}{2}$. When the velocity is 0, the diver has reached the maximum height of the dive.
Figure 3.22

EXAMPLE 11 Using the Derivative to Find Velocity

At time $t = 0$, a diver jumps from a platform diving board that is 32 feet above the water (see Figure 3.22). The position of the diver is given by

$$s(t) = -16t^2 + 16t + 32$$ Position function

where s is measured in feet and t is measured in seconds.

a. When does the diver hit the water?

b. What is the diver's velocity at impact?

Solution

a. To find the time t when the diver hits the water, let $s = 0$ and solve for t.

$$-16t^2 + 16t + 32 = 0$$ Set position function equal to 0.

$$-16(t + 1)(t - 2) = 0$$ Factor.

$$t = -1 \text{ or } 2$$ Solve for t.

Because $t \geq 0$, choose the positive value to conclude that the diver hits the water at $t = 2$ seconds.

b. The velocity at time t is given by the derivative $s'(t) = -32t + 16$. So, the velocity at time $t = 2$ is

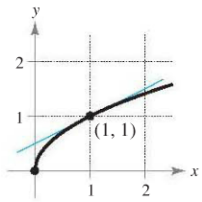
$$s'(2) = -32(2) + 16 = -48 \text{ feet per second.}$$ ■

3.2 Exercises

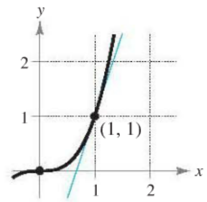
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use the graph to estimate the slope of the tangent line to $y = x^n$ at the point (1, 1). Verify your answer analytically. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

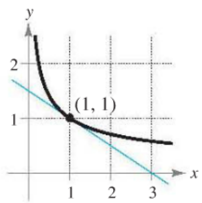
1. (a) $y = x^{1/2}$



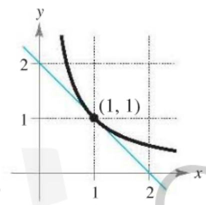
(b) $y = x^3$



2. (a) $y = x^{-1/2}$



(b) $y = x^{-1}$



In Exercises 3–26, use the rules of differentiation to find the derivative of the function.

3. $y = 12$
4. $f(x) = -9$
5. $y = x^7$
6. $y = x^{16}$
7. $y = \frac{1}{x^5}$
8. $y = \frac{1}{x^8}$
9. $f(x) = \sqrt[5]{x}$
10. $g(x) = \sqrt[6]{x}$
11. $f(x) = x + 11$
12. $g(x) = 3x - 1$
13. $f(t) = -2t^2 + 3t - 6$
14. $y = t^2 + 2t - 3$
15. $g(x) = x^2 + 4x^3$
16. $y = 8 - x^3$
17. $s(t) = t^3 + 5t^2 - 3t + 8$
18. $f(x) = 2x^3 - 4x^2 + 3x$
19. $f(x) = 6x - 5e^x$
20. $h(t) = t^3 + 2e^t$
21. $y = \frac{\pi}{2}\sin\theta - \cos\theta$
22. $g(t) = \pi\cos t$
23. $y = x^2 - \frac{1}{2}\cos x$
24. $y = 7 + \sin x$
25. $y = \frac{1}{2}e^x - 3\sin x$
26. $y = \frac{3}{4}e^x + 2\cos x$

In Exercises 27–32, complete the table.

Original Function	*Rewrite*	*Differentiate*	*Simplify*
27. $y = \frac{5}{2x^2}$			
28. $y = \frac{4}{3x^2}$			
29. $y = \frac{6}{(5x)^3}$			
30. $y = \frac{\pi}{(5x)^2}$			
31. $y = \frac{\sqrt{x}}{x}$			
32. $y = \frac{4}{x^{-3}}$			

In Exercises 33–40, find the slope of the graph of the function at the given point. Use the *derivative* feature of a graphing utility to confirm your results.

Function	*Point*
33. $f(x) = \frac{8}{x^2}$	$(2, 2)$
34. $f(t) = 3 - \frac{3}{5t}$	$\left(\frac{3}{5}, 2\right)$
35. $f(x) = -\frac{1}{2} + \frac{7}{5}x^3$	$\left(0, -\frac{1}{2}\right)$
36. $f(x) = 3(5 - x)^2$	$(5, 0)$
37. $f(\theta) = 4\sin\theta - \theta$	$(0, 0)$
38. $g(t) = -2\cos t + 5$	$(\pi, 7)$
39. $f(t) = \frac{3}{4}e^t$	$\left(0, \frac{3}{4}\right)$
40. $g(x) = -4e^x$	$(1, -4e)$

In Exercises 41–56, find the derivative of the function.

41. $g(t) = t^2 - \frac{4}{t^3}$
42. $f(x) = x + \frac{1}{x^2}$
43. $f(x) = \frac{4x^3 + 3x^2}{x}$
44. $f(x) = \frac{x^3 - 6}{x^2}$
45. $f(x) = \frac{x^3 - 3x^2 + 4}{x^2}$
46. $h(x) = \frac{2x^2 - 3x + 1}{x}$
47. $y = x(x^2 + 1)$
48. $y = 3x(6x - 5x^2)$
49. $f(x) = \sqrt{x} - 6\sqrt[3]{x}$
50. $f(x) = \sqrt[3]{x} + \sqrt[5]{x}$
51. $h(s) = s^{4/5} - s^{2/3}$
52. $f(t) = t^{2/3} - t^{1/3} + 4$
53. $f(x) = 6\sqrt{x} + 5\cos x$
54. $f(x) = \frac{2}{\sqrt[3]{x}} + 5\cos x$
55. $f(x) = x^{-2} - 2e^x$
56. $g(x) = \sqrt{x} - 3e^x$

In Exercises 57–60, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

Function	*Point*
57. $y = x^4 - x$	$(-1, 2)$
58. $f(x) = \frac{2}{\sqrt[4]{x^3}}$	$(1, 2)$
59. $g(x) = x + e^x$	$(0, 1)$
60. $h(t) = \sin t + \frac{1}{2}e^t$	$\left(\pi, \frac{1}{2}e^{\pi}\right)$