3.5 Implicit Differentiation

- **Distinguish between functions written in implicit form and explicit form.**
- **Use implicit differentiation to find the derivative of a function.**
- **Find derivatives of functions using logarithmic differentiation.**

> **EXPLORATION**
>
> ***Graphing an Implicit Equation***
> How could you use a graphing utility to sketch the graph of the equation
>
> $$x^2 - 2y^3 + 4y = 2?$$
>
> Here are two possible approaches.
>
> **a.** Solve the equation for x. Switch the roles of x and y and graph the two resulting equations. The combined graphs will show a 90° rotation of the graph of the original equation.
>
> **b.** Set the graphing utility to *parametric* mode and graph the equations
>
> $$x = -\sqrt{2t^3 - 4t + 2}$$
> $$y = t$$
>
> and
>
> $$x = \sqrt{2t^3 - 4t + 2}$$
> $$y = t.$$
>
> From either of these two approaches, can you decide whether the graph has a tangent line at the point $(0, 1)$? Explain your reasoning.

Implicit and Explicit Functions

Up to this point in the text, most functions have been expressed in **explicit form.** For example, in the equation

$$y = 3x^2 - 5 \qquad \text{Explicit form}$$

the variable y is explicitly written as a function of x. Some functions, however, are only *implied* by an equation. For instance, the function $y = 1/x$ is defined **implicitly** by the equation $xy = 1$. Suppose you were asked to find dy/dx for this equation. You could begin by writing y explicitly as a function of x and then differentiating.

Implicit Form	*Explicit Form*	*Derivative*
$xy = 1$	$y = \frac{1}{x} = x^{-1}$	$\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$

This strategy works whenever you can solve for the function explicitly. You cannot, however, use this procedure when you are unable to solve for y as a function of x. For instance, how would you find dy/dx for the equation $x^2 - 2y^3 + 4y = 2$, where it is very difficult to express y as a function of x explicitly? To do this, you can use **implicit differentiation.**

To understand how to find dy/dx implicitly, you must realize that the differentiation is taking place *with respect to* x. This means that when you differentiate terms involving x alone, you can differentiate as usual. However, when you differentiate terms involving y, you must apply the Chain Rule, because you are assuming that y is defined implicitly as a differentiable function of x.

EXAMPLE 1 Differentiating with Respect to x

a. $\frac{d}{dx}[x^3] = 3x^2$ — Variables agree: use Simple Power Rule.

(Variables agree)

b. $\frac{d}{dx}[\overbrace{y^3}^{u^n}] = \overbrace{3y^2}^{nu^{n-1}}\overbrace{\frac{dy}{dx}}^{u'}$ — Variables disagree: use Chain Rule.

(Variables disagree)

c. $\frac{d}{dx}[x + 3y] = 1 + 3\frac{dy}{dx}$ — Chain Rule: $\frac{d}{dx}[3y] = 3y'$

d. $\frac{d}{dx}[xy^2] = x\frac{d}{dx}[y^2] + y^2\frac{d}{dx}[x]$ — Product Rule

$= x\left(2y\frac{dy}{dx}\right) + y^2(1)$ — Chain Rule

$= 2xy\frac{dy}{dx} + y^2$ — Simplify. ■

Implicit Differentiation

GUIDELINES FOR IMPLICIT DIFFERENTIATION

1. Differentiate both sides of the equation *with respect to* x.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx by dividing both sides of the equation by the left-hand factor that does not contain dy/dx.

In Example 2, note that implicit differentiation can produce an expression for dy/dx that contains both x and y.

EXAMPLE 2 Implicit Differentiation

Find dy/dx given that $y^3 + y^2 - 5y - x^2 = -4$.

Solution

1. Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}[y^3 + y^2 - 5y - x^2] = \frac{d}{dx}[-4]$$

$$\frac{d}{dx}[y^3] + \frac{d}{dx}[y^2] - \frac{d}{dx}[5y] - \frac{d}{dx}[x^2] = \frac{d}{dx}[-4]$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

2. Collect the dy/dx terms on the left side of the equation.

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} = 2x$$

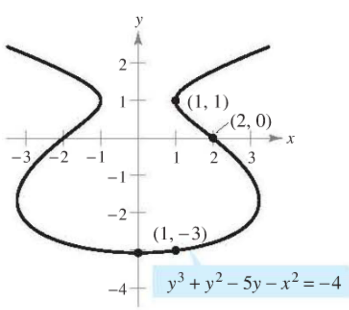
3. Factor dy/dx out of the left side of the equation.

$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

4. Solve for dy/dx by dividing by $(3y^2 + 2y - 5)$.

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$

To see how you can use an *implicit derivative*, consider the graph shown in Figure 3.27. From the graph, you can see that y is not a function of x. Even so, the derivative found in Example 2 gives a formula for the slope of the tangent line at a point on this graph. The slopes at several points on the graph are shown below the graph.



Point on Graph	*Slope of Graph*
$(2, 0)$	$-\frac{4}{5}$
$(1, -3)$	$\frac{1}{8}$
$x = 0$	0
$(1, 1)$	Undefined

The implicit equation

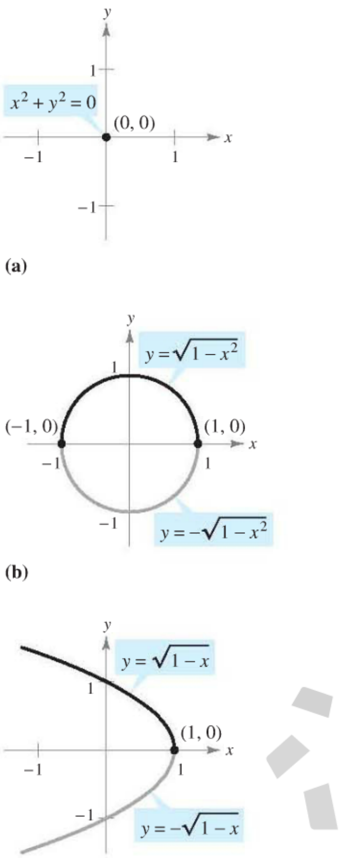
$$y^3 + y^2 - 5y - x^2 = -4$$

has the derivative

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}.$$

Figure 3.27

TECHNOLOGY With most graphing utilities, it is easy to graph an equation that explicitly represents y as a function of x. Graphing other equations, however, can require some ingenuity. For instance, to graph the equation given in Example 2, use a graphing utility, set in *parametric* mode, to graph the parametric representations $x = \sqrt{t^3 + t^2 - 5t + 4}$, $y = t$, and $x = -\sqrt{t^3 + t^2 - 5t + 4}$, $y = t$, for $-5 \le t \le 5$. How does the result compare with the graph shown in Figure 3.27?



Some graph segments can be represented by differentiable functions.
Figure 3.28

It is meaningless to solve for dy/dx in an equation that has no solution points. (For example, $x^2 + y^2 = -4$ has no solution points.) If, however, a segment of a graph can be represented by a differentiable function, dy/dx will have meaning as the slope at each point on the segment. Recall that a function is not differentiable at (1) points with vertical tangents and (2) points at which the function is not continuous.

EXAMPLE 3 Representing a Graph by Differentiable Functions

If possible, represent y as a differentiable function of x (see Figure 3.28).

a. $x^2 + y^2 = 0$ **b.** $x^2 + y^2 = 1$ **c.** $x + y^2 = 1$

Solution

a. The graph of this equation is a single point. So, the equation does not define y as a differentiable function of x.

b. The graph of this equation is the unit circle, centered at (0, 0). The upper semicircle is given by the differentiable function

$$y = \sqrt{1 - x^2}, \quad -1 < x < 1$$

and the lower semicircle is given by the differentiable function

$$y = -\sqrt{1 - x^2}, \quad -1 < x < 1.$$

At the points $(-1, 0)$ and $(1, 0)$, the slope of the graph is undefined.

c. The upper half of this parabola is given by the differentiable function

$$y = \sqrt{1 - x}, \quad x < 1$$

and the lower half of this parabola is given by the differentiable function

$$y = -\sqrt{1 - x}, \quad x < 1.$$

At the point (1, 0), the slope of the graph is undefined.

EXAMPLE 4 Finding the Slope of a Graph Implicitly

Determine the slope of the tangent line to the graph of

$$x^2 + 4y^2 = 4$$

at the point $(\sqrt{2}, -1/\sqrt{2})$. See Figure 3.29.

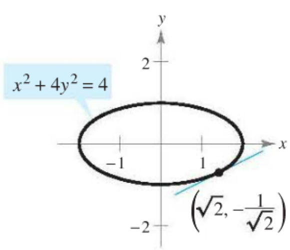


Figure 3.29

Solution

$$x^2 + 4y^2 = 4 \qquad \text{Write original equation.}$$

$$2x + 8y\frac{dy}{dx} = 0 \qquad \text{Differentiate with respect to } x.$$

$$\frac{dy}{dx} = \frac{-2x}{8y} = \frac{-x}{4y} \qquad \text{Solve for } \frac{dy}{dx}.$$

So, at $(\sqrt{2}, -1/\sqrt{2})$, the slope is

$$\frac{dy}{dx} = \frac{-\sqrt{2}}{-4/\sqrt{2}} = \frac{1}{2}. \qquad \text{Evaluate } \frac{dy}{dx} \text{ when } x = \sqrt{2} \text{ and } y = -\frac{1}{\sqrt{2}}.$$

■

NOTE To see the benefit of implicit differentiation, try doing Example 4 using the explicit function $y = -\frac{1}{2}\sqrt{4 - x^2}$. ■

EXAMPLE 5 Finding the Slope of a Graph Implicitly

Determine the slope of the graph of $3(x^2 + y^2)^2 = 100xy$ at the point (3, 1).

Solution

$$\frac{d}{dx}[3(x^2 + y^2)^2] = \frac{d}{dx}[100xy]$$

$$3(2)(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 100\left[x\frac{dy}{dx} + y(1)\right]$$

$$12y(x^2 + y^2)\frac{dy}{dx} - 100x\frac{dy}{dx} = 100y - 12x(x^2 + y^2)$$

$$[12y(x^2 + y^2) - 100x]\frac{dy}{dx} = 100y - 12x(x^2 + y^2)$$

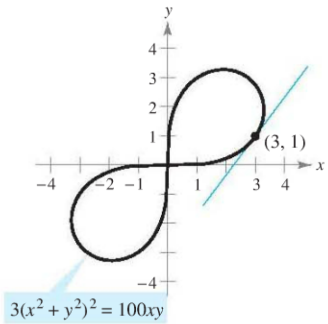
$$\frac{dy}{dx} = \frac{100y - 12x(x^2 + y^2)}{-100x + 12y(x^2 + y^2)}$$

$$= \frac{25y - 3x(x^2 + y^2)}{-25x + 3y(x^2 + y^2)}$$

At the point (3, 1), the slope of the graph is

$$\frac{dy}{dx} = \frac{25(1) - 3(3)(3^2 + 1^2)}{-25(3) + 3(1)(3^2 + 1^2)} = \frac{25 - 90}{-75 + 30} = \frac{-65}{-45} = \frac{13}{9}$$

as shown in Figure 3.30. This graph is called a **lemniscate.**



Lemniscate
Figure 3.30

EXAMPLE 6 Determining a Differentiable Function

Find dy/dx implicitly for the equation $\sin y = x$. Then find the largest interval of the form $-a < y < a$ on which y is a differentiable function of x (see Figure 3.31).

Solution

$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$$

$$\cos y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

The largest interval about the origin for which y is a differentiable function of x is $-\pi/2 < y < \pi/2$. To see this, note that $\cos y$ is positive for all y in this interval and is 0 at the endpoints. If you restrict y to the interval $-\pi/2 < y < \pi/2$, you should be able to write dy/dx explicitly as a function of x. To do this, you can use

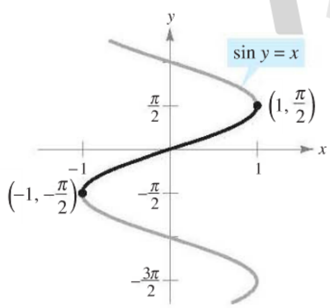
$$\cos y = \sqrt{1 - \sin^2 y}$$

$$= \sqrt{1 - x^2}, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

and conclude that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

You will study this example further when derivatives of inverse trigonometric functions are defined in Section 3.6. ■



The derivative is $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$.
Figure 3.31

The Granger Collection

ISAAC BARROW (1630–1677)

The graph in Example 8 is called the **kappa curve** because it resembles the Greek letter kappa, κ. The general solution for the tangent line to this curve was discovered by the English mathematician Isaac Barrow. Newton was Barrow's student, and they corresponded frequently regarding their work in the early development of calculus.

With implicit differentiation, the form of the derivative often can be simplified (as in Example 6) by an appropriate use of the *original* equation. A similar technique can be used to find and simplify higher-order derivatives obtained implicitly.

EXAMPLE 7 Finding the Second Derivative Implicitly

Given $x^2 + y^2 = 25$, find $\dfrac{d^2y}{dx^2}$.

Solution Differentiating each term with respect to x produces

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}.$$

Differentiating a second time with respect to x yields

$$\frac{d^2y}{dx^2} = -\frac{(y)(1) - (x)(dy/dx)}{y^2} \qquad \text{Quotient Rule}$$

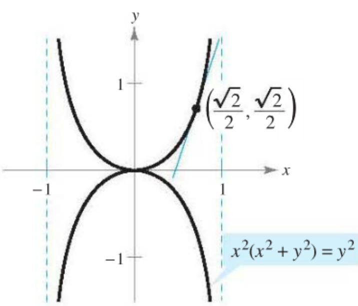
$$= -\frac{y - (x)(-x/y)}{y^2} \qquad \text{Substitute } -x/y \text{ for } \frac{dy}{dx}.$$

$$= -\frac{y^2 + x^2}{y^3} \qquad \text{Simplify.}$$

$$= -\frac{25}{y^3}. \qquad \text{Substitute 25 for } x^2 + y^2.$$

EXAMPLE 8 Finding a Tangent Line to a Graph

Find the tangent line to the graph given by $x^2(x^2 + y^2) = y^2$ at the point $(\sqrt{2}/2, \sqrt{2}/2)$, as shown in Figure 3.32.



Kappa curve
Figure 3.32

Solution By rewriting and differentiating implicitly, you obtain

$$x^4 + x^2y^2 - y^2 = 0$$

$$4x^3 + x^2\left(2y\frac{dy}{dx}\right) + 2xy^2 - 2y\frac{dy}{dx} = 0$$

$$2y(x^2 - 1)\frac{dy}{dx} = -2x(2x^2 + y^2)$$

$$\frac{dy}{dx} = \frac{x(2x^2 + y^2)}{y(1 - x^2)}.$$

At the point $(\sqrt{2}/2, \sqrt{2}/2)$, the slope is

$$\frac{dy}{dx} = \frac{(\sqrt{2}/2)[2(1/2) + (1/2)]}{(\sqrt{2}/2)[1 - (1/2)]} = \frac{3/2}{1/2} = 3$$

and the equation of the tangent line at this point is

$$y - \frac{\sqrt{2}}{2} = 3\left(x - \frac{\sqrt{2}}{2}\right)$$

$$y = 3x - \sqrt{2}.$$

Logarithmic Differentiation

On occasion, it is convenient to use logarithms as aids in differentiating nonlogarithmic functions. This procedure is called **logarithmic differentiation.**

EXAMPLE 9 Logarithmic Differentiation

Find the derivative of $y = \dfrac{(x-2)^2}{\sqrt{x^2+1}},\ x \neq 2.$

Solution Note that $y > 0$ and so $\ln y$ is defined. Begin by taking the natural logarithms of both sides of the equation. Then apply logarithmic properties and differentiate implicitly. Finally, solve for y'.

$$\ln y = \ln \frac{(x-2)^2}{\sqrt{x^2+1}} \qquad \text{Take ln of both sides.}$$

$$\ln y = 2\ln(x-2) - \frac{1}{2}\ln(x^2+1) \qquad \text{Logarithmic properties}$$

$$\frac{y'}{y} = 2\left(\frac{1}{x-2}\right) - \frac{1}{2}\left(\frac{2x}{x^2+1}\right) \qquad \text{Differentiate.}$$

$$= \frac{x^2+2x+2}{(x-2)(x^2+1)} \qquad \text{Simplify.}$$

$$y' = y\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] \qquad \text{Solve for } y'.$$

$$= \frac{(x-2)^2}{\sqrt{x^2+1}}\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] \qquad \text{Substitute for } y.$$

$$= \frac{(x-2)(x^2+2x+2)}{(x^2+1)^{3/2}} \qquad \text{Simplify.}$$

3.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–22, find dy/dx by implicit differentiation.

1. $x^2 + y^2 = 9$
2. $x^2 - y^2 = 25$
3. $x^{1/2} + y^{1/2} = 16$
4. $x^3 + y^3 = 64$
5. $x^3 - xy + y^2 = 7$
6. $x^2y + y^2x = -3$
7. $xe^y - 10x + 3y = 0$
8. $e^{xy} + x^2 - y^2 = 10$
9. $x^3y^3 - y = x$
10. $\sqrt{xy} = x^2y + 1$
11. $x^3 - 3x^2y + 2xy^2 = 12$
12. $4\cos x \sin y = 1$
13. $\sin x + 2\cos 2y = 1$
14. $(\sin \pi x + \cos \pi y)^2 = 2$
15. $\sin x = x(1 + \tan y)$
16. $\cot y = x - y$
17. $y = \sin(xy)$
18. $x = \sec \dfrac{1}{y}$
19. $x^2 - 3\ln y + y^2 = 10$
20. $\ln xy + 5x = 30$
21. $4x^3 + \ln y^2 + 2y = 2x$
22. $4xy + \ln x^2y = 7$

In Exercises 23–26, (a) find two explicit functions by solving the equation for y in terms of x, (b) sketch the graph of the equation and label the parts given by the corresponding explicit functions, (c) differentiate the explicit functions, and (d) find dy/dx implicitly and show that the result is equivalent to that of part (c).

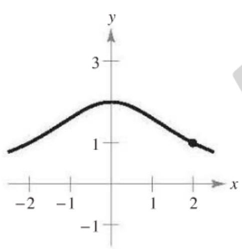
23. $x^2 + y^2 = 64$
24. $x^2 + y^2 - 4x + 6y + 9 = 0$
25. $16x^2 + 25y^2 = 400$
26. $16y^2 - x^2 = 16$

In Exercises 27–36, find dy/dx by implicit differentiation and evaluate the derivative at the given point.

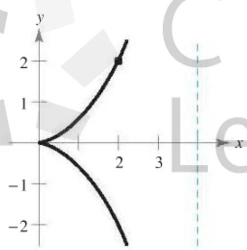
Equation	*Point*
27. $xy = 6$	$(-6, -1)$
28. $x^3 - y^2 = 0$	$(1, 1)$
29. $y^2 = \dfrac{x^2 - 49}{x^2 + 49}$	$(7, 0)$
30. $(x + y)^3 = x^3 + y^3$	$(-1, 1)$
31. $x^{2/3} + y^{2/3} = 5$	$(8, 1)$
32. $x^3 + y^3 = 6xy - 1$	$(2, 3)$
33. $\tan(x + y) = x$	$(0, 0)$
34. $x \cos y = 1$	$\left(2, \dfrac{\pi}{3}\right)$
35. $3e^{xy} - x = 0$	$(3, 0)$
36. $y^2 = \ln x$	$(e, 1)$

Famous Curves **In Exercises 37–40, find the slope of the tangent line to the graph at the given point.**

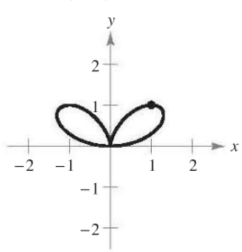
37. Witch of Agnesi:
$(x^2 + 4)y = 8$
Point: $(2, 1)$



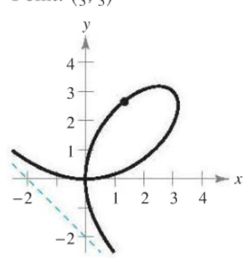
38. Cissoid:
$(4 - x)y^2 = x^3$
Point: $(2, 2)$



39. Bifolium:
$(x^2 + y^2)^2 = 4x^2y$
Point: $(1, 1)$



40. Folium of Descartes:
$x^3 + y^3 - 6xy = 0$
Point: $\left(\frac{4}{3}, \frac{8}{3}\right)$



In Exercises 41–44, use implicit differentiation to find an equation of the tangent line to the graph at the given point.

41. $4xy = 9, \quad \left(1, \frac{9}{4}\right)$

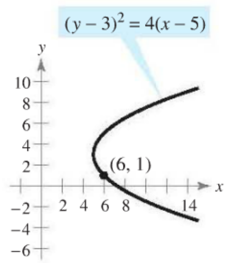
42. $x^2 + xy + y^2 = 4, \quad (2, 0)$

43. $x + y - 1 = \ln(x^2 + y^2), \quad (1, 0)$

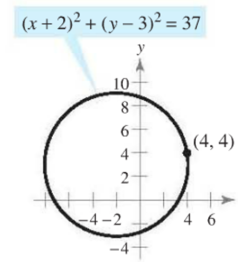
44. $y^2 + \ln xy = 2, \quad (e, 1)$

Famous Curves **In Exercises 45–52, find an equation of the tangent line to the graph at the given point. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

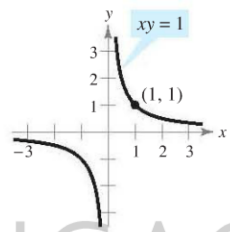
45. Parabola



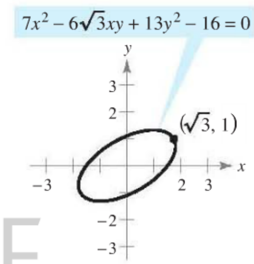
46. Circle



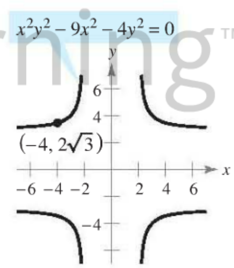
47. Rotated hyperbola



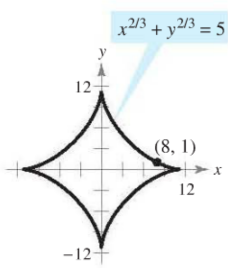
48. Rotated ellipse



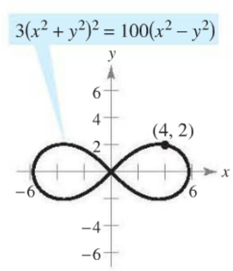
49. Cruciform



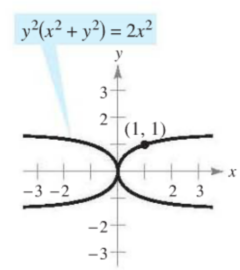
50. Astroid



51. Lemniscate



52. Kappa curve



53. (a) Use implicit differentiation to find an equation of the tangent line to the ellipse $\dfrac{x^2}{2} + \dfrac{y^2}{8} = 1$ at $(1, 2)$.

(b) Show that the equation of the tangent line to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at (x_0, y_0) is $\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} = 1$.

54. (a) Use implicit differentiation to find an equation of the tangent line to the hyperbola $\frac{x^2}{6} - \frac{y^2}{8} = 1$ at $(3, -2)$.

(b) Show that the equation of the tangent line to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_0, y_0) is $\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$.

In Exercises 55 and 56, find dy/dx implicitly and find the largest interval of the form $-a < y < a$ or $0 < y < a$ such that y is a differentiable function of x. Write dy/dx as a function of x.

55. $\tan y = x$

56. $\cos y = x$

In Exercises 57–62, find d^2y/dx^2 in terms of x and y.

57. $x^2 + y^2 = 4$

58. $x^2y^2 - 2x = 3$

59. $x^2 - y^2 = 36$

60. $1 - xy = x - y$

61. $y^2 = x^3$

62. $y^2 = 10x$

In Exercises 63 and 64, use a graphing utility to graph the equation. Find an equation of the tangent line to the graph at the given point and graph the tangent line in the same viewing window.

63. $\sqrt{x} + \sqrt{y} = 5, \quad (9, 4)$

64. $y^2 = \frac{x-1}{x^2+1}, \quad \left(2, \frac{\sqrt{5}}{5}\right)$

In Exercises 65 and 66, find equations of the tangent line and normal line to the circle at each given point. (The *normal line* at a point is perpendicular to the tangent line at the point.) Use a graphing utility to graph the equation, tangent line, and normal line.

65. $x^2 + y^2 = 25$
$(4, 3), (-3, 4)$

66. $x^2 + y^2 = 36$
$(6, 0), (5, \sqrt{11})$

67. Show that the normal line at any point on the circle $x^2 + y^2 = r^2$ passes through the origin.

68. Two circles of radius 4 are tangent to the graph of $y^2 = 4x$ at the point $(1, 2)$. Find equations of these two circles.

In Exercises 69 and 70, find the points at which the graph of the equation has a vertical or horizontal tangent line.

69. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

70. $4x^2 + y^2 - 8x + 4y + 4 = 0$

In Exercises 71–82, find dy/dx using logarithmic differentiation.

71. $y = x\sqrt{x^2 + 1}, \quad x > 0$

72. $y = \sqrt{x^2(x + 1)(x + 2)}, \quad x > 0$

73. $y = \frac{x^2\sqrt{3x - 2}}{(x + 1)^2}, \quad x > \frac{2}{3}$

74. $y = \sqrt{\frac{x^2 - 1}{x^2 + 1}}, \quad x > 1$

75. $y = \frac{x(x - 1)^{3/2}}{\sqrt{x + 1}}, \quad x > 1$

76. $y = \frac{(x + 1)(x - 2)}{(x - 1)(x + 2)}, \quad x > 2$

77. $y = x^{2/x}, \quad x > 0$

78. $y = x^{x-1}, \quad x > 0$

79. $y = (x - 2)^{x+1}, \quad x > 2$

80. $y = (1 + x)^{1/x}, \quad x > 0$

81. $y = x^{\ln x}, \quad x > 0$

82. $y = (\ln x)^{\ln x}, \quad x > 1$

Orthogonal Trajectories **In Exercises 83–86, use a graphing utility to graph the intersecting graphs of the equations and show that they are orthogonal. [Two graphs are *orthogonal* if at their point(s) of intersection their tangent lines are perpendicular to each other.]**

83. $2x^2 + y^2 = 6$
$y^2 = 4x$

84. $y^2 = x^3$
$2x^2 + 3y^2 = 5$

85. $x + y = 0$
$x = \sin y$

86. $x^3 = 3(y - 1)$
$x(3y - 29) = 3$

Orthogonal Trajectories **In Exercises 87 and 88, verify that the two families of curves are orthogonal, where C and K are real numbers. Use a graphing utility to graph the two families for two values of C and two values of K.**

87. $xy = C, \quad x^2 - y^2 = K$

88. $x^2 + y^2 = C^2, \quad y = Kx$

In Exercises 89–92, differentiate (a) with respect to x (y is a function of x) and (b) with respect to t (x and y are functions of t).

89. $2y^2 - 3x^4 = 0$

90. $x^2 - 3xy^2 + y^3 = 10$

91. $\cos \pi y - 3 \sin \pi x = 1$

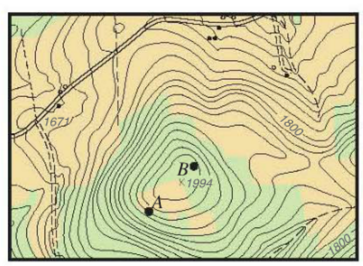
92. $4 \sin x \cos y = 1$

WRITING ABOUT CONCEPTS

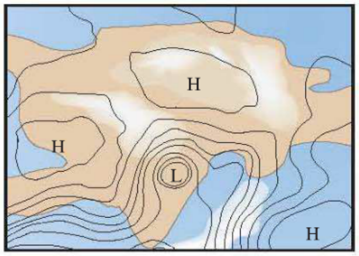
93. Describe the difference between the explicit form of a function and an implicit equation. Give an example of each.

94. In your own words, state the guidelines for implicit differentiation.

95. ***Orthogonal Trajectories*** The figure below shows the topographic map carried by a group of hikers. The hikers are in a wooded area on top of the hill shown on the map and they decide to follow a path of steepest descent (orthogonal trajectories to the contours on the map). Draw their routes if they start from point A and if they start from point B. If their goal is to reach the road along the top of the map, which starting point should they use? To print an enlarged copy of the map, go to the website *www.mathgraphs.com*.



96. ***Weather Map*** The weather map shows several *isobars*—curves that represent areas of constant air pressure. Three high pressures H and one low pressure L are shown on the map. Given that wind speed is greatest along the orthogonal trajectories of the isobars, use the map to determine the areas having high wind speed.



97. Consider the equation $x^4 = 4(4x^2 - y^2)$.

(a) Use a graphing utility to graph the equation.

(b) Find and graph the four tangent lines to the curve for $y = 3$.

(c) Find the exact coordinates of the point of intersection of the two tangent lines in the first quadrant.

CAPSTONE

98. Determine if the statement is true. If it is false, explain why and correct it. For each statement, assume y is a function of x.

(a) $\dfrac{d}{dx}\cos(x^2) = -2x\sin(x^2)$ (b) $\dfrac{d}{dy}\cos(y^2) = 2y\sin(y^2)$

(c) $\dfrac{d}{dx}\cos(y^2) = -2y\sin(y^2)$

99. Let L be any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$. Show that the sum of the x- and y-intercepts of L is c.

100. (a) Prove (Theorem 3.3) that $d/dx[x^n] = nx^{n-1}$ for the case in which n is a rational number. (*Hint:* Write $y = x^{p/q}$ in the form $y^q = x^p$ and differentiate implicitly. Assume that p and q are integers, where $q > 0$.)

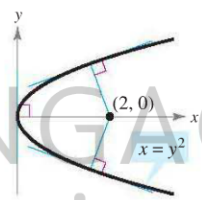
(b) Prove part (a) for the case in which n is an irrational number. (*Hint:* Let $y = x^r$, where r is a real number, and use logarithmic differentiation.)

101. ***Slope*** Find all points on the circle $x^2 + y^2 = 100$ where the slope is $\frac{3}{4}$.

102. ***Horizontal Tangent Line*** Determine the point(s) at which the graph of $y^4 = y^2 - x^2$ has a horizontal tangent line.

103. ***Tangent Lines*** Find equations of both tangent lines to the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ that passes through the point $(4, 0)$.

104. ***Normals to a Parabola*** The graph shows the normal lines from the point $(2, 0)$ to the graph of the parabola $x = y^2$. How many normal lines are there from the point $(x_0, 0)$ to the graph of the parabola if (a) $x_0 = \frac{1}{4}$, (b) $x_0 = \frac{1}{2}$, and (c) $x_0 = 1$? For what value of x_0 are two of the normal lines perpendicular to each other?



105. ***Normal Lines*** (a) Find an equation of the normal line to the ellipse $\dfrac{x^2}{32} + \dfrac{y^2}{8} = 1$ at the point $(4, 2)$. (b) Use a graphing utility to graph the ellipse and the normal line. (c) At what other point does the normal line intersect the ellipse?

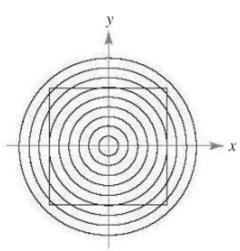
SECTION PROJECT

Optical Illusions

In each graph below, an optical illusion is created by having lines intersect a family of curves. In each case, the lines appear to be curved. Find the value of dy/dx for the given values of x and y.

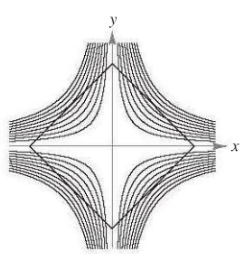
(a) Circles: $x^2 + y^2 = C^2$

$x = 3, y = 4, C = 5$



(b) Hyperbolas: $xy = C$

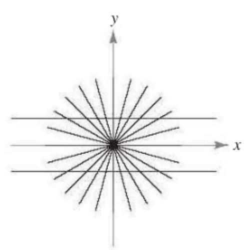
$x = 1, y = 4, C = 4$



(c) Lines: $ax = by$

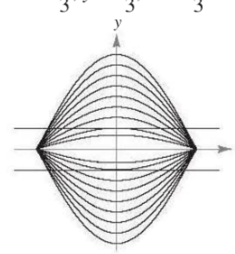
$x = \sqrt{3}, y = 3,$

$a = \sqrt{3}, b = 1$



(d) Cosine curves: $y = C\cos x$

$x = \dfrac{\pi}{3}, y = \dfrac{1}{3}, C = \dfrac{2}{3}$



FOR FURTHER INFORMATION For more information on the mathematics of optical illusions, see the article "Descriptive Models for Perception of Optical Illusions" by David A. Smith in *The UMAP Journal*.